

ABOUT THE MULTIFRACTAL NATURE OF CANTOR'S BIJECTION

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ABSTRACT. In this note, we investigate the regularity of Cantor's one-to-one mapping between the irrational numbers of the unit interval and the irrational numbers of the unit square. In particular, we explore the fractal nature of this map by showing that its Hölder regularity lies between 0.35 and 0.72 almost everywhere (with respect to the Lebesgue measure).

MSC: 26A30, 11K50.

1. INTRODUCTION

In 1878 [1], Cantor proved that there exists a one-to-one correspondence between the points of the unit line segment $[0, 1]$ and the points of the unit square $[0, 1]^2$ (repeated application of this result gives a bijective correspondence between $[0, 1]$ and $[0, 1]^n$, where n is a natural number). About this discovery he wrote to Dedekind: “Je le vois, mais je ne le crois pas !” (“I see it, but I don't believe it!”) [11, 2]. Since this application is defined via continued fractions, it is very hard to have any intuition about its regularity. When looking at its definition or at the graphical representation of each component (given for the first time here), it is not hard to convince oneself that the behavior of such a function is necessarily “erratic”; however, its (Hölder-)regularity has never been considered.

The set of the natural numbers is denoted by \mathbb{N} (and does not contain 0). We set $E = [0, 1]$, denote by D the rational numbers of E and set $I = E \setminus D$. The set of the (infinite) sequences of natural numbers is denoted $\mathbb{N}^{\mathbb{N}}$; since this space is a countable product of metric spaces, if $\mathbf{a} = (a_j)_{j \in \mathbb{N}}$ and $\mathbf{b} = (b_j)_{j \in \mathbb{N}}$ are two elements of $\mathbb{N}^{\mathbb{N}}$, we define the usual distance

$$d(\mathbf{a}, \mathbf{b}) = \sum_{j=1}^{\infty} 2^{-j} \frac{|a_j - b_j|}{|a_j - b_j| + 1}.$$

We will implicitly consider that $\mathbb{N}^{\mathbb{N}}$ is equipped with this distance, while E , D and I are endowed with the Euclidean distance.

Remark 1. Considering \mathbf{a} and \mathbf{b} as two infinite words on the alphabet \mathbb{N} [7], one can also use the following ultrametric distance on $\mathbb{N}^{\mathbb{N}}$: if $\mathbf{a} = (a_j)_{j \in \mathbb{N}}$ and $\mathbf{b} = (b_j)_{j \in \mathbb{N}}$ both belong to $\mathbb{N}^{\mathbb{N}}$, let $\mathbf{a} \wedge \mathbf{b}$ denote the longest common prefix of \mathbf{a} and \mathbf{b} , so that the length $|\mathbf{a} \wedge \mathbf{b}|$ of this prefix is equal to the lowest natural number j such that $a_j \neq b_j$ minus 1. A distance between \mathbf{a} and \mathbf{b} is given by

$$d'(\mathbf{a}, \mathbf{b}) = \begin{cases} 0 & \text{if } \mathbf{a} = \mathbf{b} \\ 2^{-|\mathbf{a} \wedge \mathbf{b}|} & \text{if } \mathbf{a} \neq \mathbf{b} \end{cases}.$$

The following relations hold:

$$2^{-2}d \leq d' < d.$$

For the sake of completeness, let us recall the following result.

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Proposition 1. *The space $\mathbb{N}^{\mathbb{N}}$ (endowed with the distance defined above) is a separable complete metric space.*

Proof. If $\mathbb{N}_n^{\mathbb{N}}$ denotes the set $\{\mathbf{a} = (a_j)_{j \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}} : a_j = 1 \forall j > n\}$ ($n \in \mathbb{N}$), one directly checks that $\cup_{n \in \mathbb{N}} \mathbb{N}_n^{\mathbb{N}}$ is dense in $\mathbb{N}^{\mathbb{N}}$. Moreover, if \mathbf{a}_j is a Cauchy sequence of $\mathbb{N}^{\mathbb{N}}$, there exists a subsequence \mathbf{b}_j such that $d(\mathbf{b}_j, \mathbf{b}_{j+1}) < 2^{-j}$ for any $j \in \mathbb{N}$. One easily checks that \mathbf{b}_j converges to $\mathbf{a}_0 \in \mathbb{N}^{\mathbb{N}}$ as j tends to infinity, where $a_{0,k} = b_{k,k}$ ($k \in \mathbb{N}$). \square

In this note, we first recall the construction of Cantor's bijection between I and I^2 based on continued fractions and give, as far as we know for the first time, a graphical representation of the two components of this map. We then construct an homeomorphism between I and $\mathbb{N}^{\mathbb{N}}$ to show that Cantor's bijection between I and I^2 is an homeomorphism and that any extension of this mapping to E is necessarily discontinuous at every rational number. We also investigate the multifractal nature of this function. It is well known that most of the "historical" space filling functions are monoHölder with Hölder exponent equal to $1/2$ [4, 5]; here we show that for Cantor's bijection, almost every point of I (with respect to the Lebesgue measure) is associated to an Hölder exponent which belongs to an interval containing $1/2$ (more precisely, this interval is bounded by 0.35 and 0.72). All the obtained results strongly rely on the theory of the continued fractions (see e.g. [6]).

2. DEFINITIONS

2.1. Continued fractions. Let us first recall the basic facts about the continued fractions [6]. Here, we state the results for E , but they can be easily extended to the whole real line.

Let $\mathbf{a} = (a_j)_{j \in \{1, \dots, n\}}$ be a finite sequence of positive real numbers ($n \in \mathbb{N}$); the expression $[a_1, \dots, a_n]$ is recursively defined as follows:

$$[a_1] = 1/a_1 \quad \text{and} \quad [a_1, \dots, a_m] = \frac{1}{a_1 + [a_2, \dots, a_m]},$$

for any $m \in \{2, \dots, n\}$. If $\mathbf{a} \in \mathbb{N}^n$, we say that $[a_1, \dots, a_n]$ is a (simple) finite continued fraction.

Proposition 2. *For any $\mathbf{a} \in \mathbb{N}^n$ ($n \in \mathbb{N}$), $[a_1, \dots, a_n]$ belongs to D . Conversely, for any $x \in D$, there exists a natural number n and a sequence $\mathbf{a} \in \mathbb{N}^n$ such that $x = [a_1, \dots, a_n]$.*

The representation of a rational number as a continued fraction is not unique, as shown by the following remark; this will be used in the proof of Proposition 11.

Remark 2. If $\mathbf{a} \in \mathbb{N}^n$ ($n \in \mathbb{N}$) is such that $a_n > 1$, one has

$$[a_1, \dots, a_n] = [a_1, \dots, a_n - 1, 1].$$

Let us now define the notion of convergent. For $\mathbf{a} \in \mathbb{N}^n$ ($n \in \mathbb{N}$) and each integer $j \in \{-1, \dots, n\}$, let us define $p_j(\mathbf{a})$ and $q_j(\mathbf{a})$ by setting $p_{-1}(\mathbf{a}) = 1$, $q_{-1}(\mathbf{a}) = 0$, $p_0(\mathbf{a}) = 0$, $q_0(\mathbf{a}) = 1$ and, for $j \in \mathbb{N}$,

$$\begin{cases} p_j(\mathbf{a}) = a_j p_{j-1}(\mathbf{a}) + p_{j-2}(\mathbf{a}) \\ q_j(\mathbf{a}) = a_j q_{j-1}(\mathbf{a}) + q_{j-2}(\mathbf{a}) \end{cases}.$$

The quotient $p_j(\mathbf{a})/q_j(\mathbf{a})$ is called the convergent of order j of \mathbf{a} . They are intimately related to the continued fractions.

Proposition 3. *Let $[a_1, \dots, a_n]$ ($n \in \mathbb{N}$) be a continued fraction; the section $[a_1, \dots, a_j]$, with $j \leq n$, is equal to $p_j(\mathbf{a})/q_j(\mathbf{a})$. Furthermore, we have, for any $j \geq 1$,*

$$q_j(\mathbf{a})p_{j-1}(\mathbf{a}) - p_j(\mathbf{a})q_{j-1}(\mathbf{a}) = (-1)^j,$$

and, for any $j \geq 2$,

$$q_j(\mathbf{a})p_{j-2}(\mathbf{a}) - p_j(\mathbf{a})q_{j-2}(\mathbf{a}) = (-1)^{j-1}a_j.$$

As a consequence, one has, for any $j \geq 2$,

$$\frac{p_{j-1}(\mathbf{a})}{q_{j-1}(\mathbf{a})} - \frac{p_j(\mathbf{a})}{q_j(\mathbf{a})} = \frac{(-1)^j}{q_j(\mathbf{a})q_{j-1}(\mathbf{a})},$$

and, for any $j \geq 3$,

$$\frac{p_{j-2}(\mathbf{a})}{q_{j-2}(\mathbf{a})} - \frac{p_j(\mathbf{a})}{q_j(\mathbf{a})} = \frac{(-1)^{j-1}a_j}{q_j(\mathbf{a})q_{j-2}(\mathbf{a})}.$$

Of course, one can define the numbers $p_j(\mathbf{a})$ and $q_j(\mathbf{a})$ for an element \mathbf{a} of $\mathbb{N}^{\mathbb{N}}$. The convergents allow to introduce the notion of infinite continued fraction, thanks to the following trivial result.

Corollary 4. *For any $\mathbf{a} \in \mathbb{N}^{\mathbb{N}}$, the sequences*

$$\left(\frac{p_{2j-1}(\mathbf{a})}{q_{2j-1}(\mathbf{a})}\right)_{j \in \mathbb{N}} \quad \text{and} \quad \left(\frac{p_{2j}(\mathbf{a})}{q_{2j}(\mathbf{a})}\right)_{j \in \mathbb{N}}$$

are two adjacent sequences, $p_{2j}(\mathbf{a})/q_{2j}(\mathbf{a})$ being the increasing one.

This shows that for any $\mathbf{a} \in \mathbb{N}^{\mathbb{N}}$, the sequence $x_j = [a_1, \dots, a_j]$ converges. The limit is called an infinite continued fraction and is denoted $[a_1, \dots]$. If the real number $x \in E$ is equal to $[a_1, \dots]$, we say that $[a_1, \dots]$ is a continued fraction corresponding to x . The following result states that the continued fraction is an instrument for representing the real numbers (of E).

Theorem 5. *We have $x \in I$ if and only if there exists an infinite continued fraction corresponding to x ; moreover, this infinite continued fraction is unique.*

Proposition 6. *If $x \in E$ can be written as $x = [a_1, \dots, a_n, r_{n+1}]$, with $a_1, \dots, a_n \in \mathbb{N}$ and $r_{n+1} \in [1, \infty)$, the following relation holds:*

$$x = \frac{p_n(\mathbf{a})r_{n+1} + p_{n-1}(\mathbf{a})}{q_n(\mathbf{a})r_{n+1} + q_{n-1}(\mathbf{a})},$$

with $\mathbf{a} = (a_j)_{j \in \{1, \dots, n\}}$.

A sequence $\mathbf{a} \in \mathbb{N}^{\mathbb{N}}$ is ultimately periodic of period $k \in \mathbb{N}$ if there exists J such that $a_{j+k} = a_j$ for any $j \geq J$. In this case, the corresponding continued fraction $[a_1, \dots]$ is also called ultimately periodic of period k . The quadratic numbers (of E) are characterized by their corresponding continued fractions.

Theorem 7. *An element of I is a quadratic number if and only if the corresponding continued fraction is ultimately periodic.*

If \mathbf{a} is an element of $\mathbb{N}^{\mathbb{N}}$ or \mathbb{N}^n ($n \in \mathbb{N}$), we will sometimes simply write $[\mathbf{a}]$ instead of $[a_1, \dots]$ or $[a_1, \dots, a_n]$ respectively.

Let us now give a brief introduction to the notion of the metric theory of continued fractions. Since, for any $\mathbf{a} \in \mathbb{N}^{\mathbb{N}}$, $[\mathbf{a}]$ corresponds to an irrational number $x \in I$, one can consider, for each $j \in \mathbb{N}$, the term a_j as a function of x : $a_j = a_j(x)$. Let us fix $j \in \mathbb{N}$ and write $x = [a_1, \dots, a_{j-1}, r_j]$, with $r_j \in [1, \infty)$. It is easy to check that for any $k \in \mathbb{N}$, we have, if j is odd,

$$a_j = k \quad \text{if and only if} \quad \frac{1}{k+1} < r_j \leq \frac{1}{k}$$

and, if j is even,

$$a_j = k \quad \text{if and only if} \quad k \leq r_j < k+1.$$

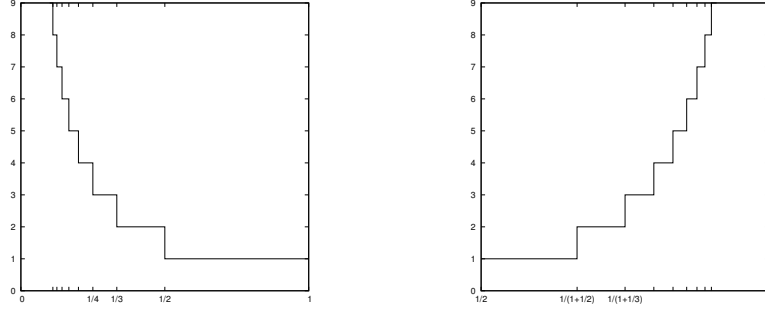


FIGURE 1. The functions $x \mapsto a_1(x)$ (left panel) and $x \mapsto a_2(x)$ if $a_1(x) = 1$ (right panel). This illustrates the fact that $I_1(x)$ is partitioned into a denumerably infinite number of intervals of rank 2; in this case, $I_2(x) \subset [1/2, 1] \cap I$, since $a_1(x) = 1$ if and only if $x \in [1/2, 1] \cap I$.

For any $j \in \mathbb{N}$, a_j is thus a piecewise constant function. Moreover, a_j is non-increasing if j is odd and non-decreasing if j is even. The functions a_1 and a_2 are represented in Figure 1. Let $x = [\mathbf{a}]$ be an irrational number; for $n \in \mathbb{N}$, we set

$$I_n(x) = \{y = [\mathbf{b}] \in I : b_j = a_j \text{ if } j \in \{1, \dots, n\}\}.$$

We will say that $I_n(x)$ is an interval of rank n . For any $n \in \mathbb{N}$, $I_n(x)$ is an irrational subinterval of I , $I_{n+1}(x) \subset I_n(x)$ and $\lim_n I_n(x) = \{x\}$. Indeed, using Proposition 6 with $r_{n+1} = 1$ and $r_{n+1} \rightarrow \infty$, one gets

$$I_n(x) = \left(\frac{p_n(\mathbf{a})}{q_n(\mathbf{a})}, \frac{p_n(\mathbf{a}) + p_{n-1}(\mathbf{a})}{q_n(\mathbf{a}) + q_{n-1}(\mathbf{a})} \right) \cap I,$$

if n is even (if n is odd, the endpoints of the interval are reversed). Every interval of rank n is partitioned into a denumerably infinite number of intervals of rank $n+1$. We will denote by $|I_n(x)|$ the Lebesgue measure of $I_n(x)$. One has, using Proposition 3,

$$|I_n(x)| = \frac{1}{q_n(\mathbf{a})(q_n(\mathbf{a}) + q_{n-1}(\mathbf{a}))}.$$

2.2. Cantor's bijection. Cantor's bijection on I (see [1]) is a one-to-one mapping between I and I^2 . If $x \in I$, let $[a_1, \dots]$ be the corresponding continued fraction and define the applications f_1 and f_2 as follows:

$$f_1(x) = [a_1, a_3, \dots, a_{2j+1}, \dots] \quad \text{and} \quad f_2(x) = [a_2, a_4, \dots, a_{2j}, \dots].$$

These applications are represented in Figure 2.

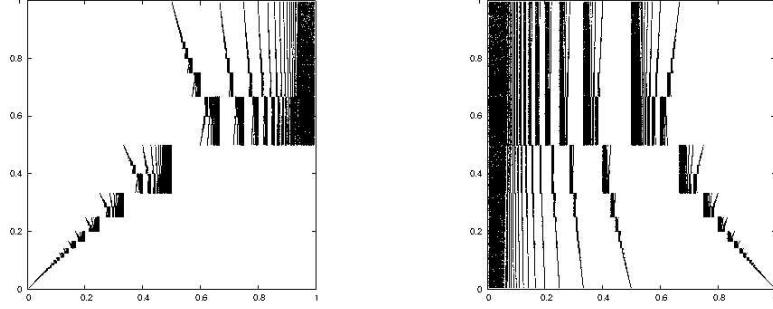
Theorem 5 implies that the application

$$f : I \rightarrow I^2 \quad x \mapsto (f_1(x), f_2(x))$$

is a one-to-one mapping. If Q denotes the quadratic numbers of I , f is a one-to-one mapping between Q to Q^2 . Since the cardinals of E and I are equal, f can be extended to a one-to-one mapping from E to E^2 .

One can already show that Cantor's bijection is continuous. However, we will be more precise in the next section, using simpler arguments.

Remark 3. For any $n \in \mathbb{N}$ and any $x \in I$, f_1 maps the interval $I_n(x)$ to $I_m(f_1(x))$, where $m = n/2$ if n is even and $m = (n+1)/2$ if n is odd. This indeed shows that f_1 is a continuous function; obviously, the same argument can be applied to f_2 .


 FIGURE 2. The functions f_1 (left panel) and f_2 (right panel).

3. CONTINUITY OF CANTOR'S BIJECTION ON I

Let $x \in I$; we write $\varphi(x) = \mathbf{a}$ if $\mathbf{a} \in \mathbb{N}^{\mathbb{N}}$ satisfies $x = [\mathbf{a}]$. For any $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the floor function and $\lceil x \rceil$ the ceil function: $\lfloor x \rfloor = \sup\{k \in \mathbb{Z} : k \leq x\}$, $\lceil x \rceil = \inf\{k \in \mathbb{Z} : x \leq k\}$.

Proposition 8. *The application φ is an homeomorphism between I and $\mathbb{N}^{\mathbb{N}}$.*

Proof. Let x_j be a sequence on I that converges to $x_0 \in I$. The fact that $\varphi(x_j)$ converges to $\varphi(x_0)$ is a direct consequence of Euclid's algorithm, but it is even simpler when one has the metric theory of continued fractions at one's disposal. For any $n \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $j \geq J$ implies $x_j \in I_n(x_0)$, which is sufficient.

Now let \mathbf{a}_j be a sequence on $\mathbb{N}^{\mathbb{N}}$ that converges to $\mathbf{a}_0 \in \mathbb{N}^{\mathbb{N}}$ and set $x_j = [a_{j,1}, \dots]$, $x_0 = [a_{0,1}, \dots]$. For $\varepsilon > 0$, let $n \in \mathbb{N}$ such that

$$q_n(\mathbf{a}_0)(q_n(\mathbf{a}_0) + q_{n-1}(\mathbf{a}_0)) > \frac{1}{\varepsilon}.$$

Since there exists $J \in \mathbb{N}$ such that $x_j \in I_n(x_0)$ whenever $j \geq J$, one has

$$|x_0 - x_j| \leq |I_n(x_0)| < \varepsilon$$

for such indexes. One can avoid the use of the metric theory of continued fractions using Proposition 3: for $\varepsilon > 0$, let $k \in \mathbb{N}$ such that $q_k(\mathbf{a}_0) > \sqrt{2/\varepsilon}$. We have

$$\begin{aligned} |x_0 - x_j| &\leq |x_0 - [a_{0,1}, \dots, a_{0,k}]| + |[a_{0,1}, \dots, a_{0,k}] - [a_{j,1}, \dots, a_{j,k}]| \\ &\quad + |[a_{j,1}, \dots, a_{j,k}] - x_j| \\ &\leq \frac{1}{q_k^2(\mathbf{a}_0)} + \frac{1}{q_k^2(\mathbf{a}_j)} = \frac{2}{q_k^2(\mathbf{a}_0)}, \end{aligned}$$

for j sufficiently large, which is sufficient to conclude. \square

Remark 4. We obviously have $[\cdot] = \varphi^{-1}$ on $\mathbb{N}^{\mathbb{N}}$.

Since $(\mathbb{N}^{\mathbb{N}}, d)$ is a separable complete metric space, we have reobtained the following well-known result.

Corollary 9. *The space I is a Polish space.*

Proposition 10. *Cantor's bijection f is an homeomorphism between I and I^2 .*

Proof. This is trivial since the application

$$\psi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}} \quad (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{c},$$

where

$$c_j = \begin{cases} a_{(j+1)/2} & \text{if } j \text{ is odd} \\ b_{j/2} & \text{if } j \text{ is even} \end{cases}$$

is an homeomorphism. \square

Netto's theorem [10] guarantees that such a function f can not be extended to a continuous function from E to E^2 . The following result gives additional informations.

Proposition 11. *Any extension of Cantor's bijection to E is discontinuous at any rational number.*

Proof. Let $x \in D$; there exists $k \in \mathbb{N}$ and $\mathbf{a} \in \mathbb{N}^k$ with $a_k > 1$ such that

$$x = [a_1, \dots, a_k] = [a_1, \dots, a_k - 1, 1].$$

Let $\mathbf{b} \in \mathbb{N}^{\mathbb{N}}$ and set $x_j = [a_1, \dots, a_k, r_j]$, $y_j = [a_1, \dots, a_k - 1, 1, r_j]$ with $r_j = j + [\mathbf{b}]$. Both the sequences x_j and y_j converge to x and it is easy to check that $\lim_j f(x_j) \neq \lim_j f(y_j)$. \square

4. HÖLDER REGULARITY OF CANTOR'S BIJECTION

In this section, we give some preliminary results about the Hölder regularity (see e.g. [3] and references therein) of Cantor's bijection.

Let $\alpha \in [0, 1]$; a continuous and bounded real function g defined on $A \subset \mathbb{R}$ belongs to the Hölder space $\Lambda^\alpha(x)$ with $x \in A$ if there exists a constant $C > 0$ such that

$$|g(x) - g(y)| \leq C|x - y|^\alpha,$$

for any $y \in A$. The Hölder exponent $h_g(x)$ of g at x is defined as follows:

$$h_g(x) = \sup\{\alpha \in [0, 1] : g \in \Lambda^\alpha(x)\}.$$

If $h_g(x) < 1$, then g is not differentiable at x .

Let us now state our main result.

Theorem 12. *Let $x = [\mathbf{a}]$ be an element of I and $y \in I_n(x) \setminus I_{n+1}(x)$. One has*

$$\frac{\frac{1}{n} \sum_{j=1}^{\lceil n/2 \rceil} \log a_{2j-1}}{\frac{1}{n} \sum_{j=1}^{n+3} \log(a_j + 1) + \frac{1}{n} C_1(n)} \leq \frac{\log |f_1(x) - f_1(y)|}{\log |x - y|}$$

and

$$\frac{\log |f_1(x) - f_1(y)|}{\log |x - y|} \leq \frac{\frac{1}{n} \sum_{j=1}^{\lceil n/2 \rceil + 3} \log(a_{2j-1} + 1) + \frac{1}{2n} C_2(n)}{\frac{1}{n} \sum_{j=1}^n \log a_j},$$

where

$$C_1(n) = \frac{\log 2}{2} + \log \max\left(\frac{a_{n+2} + 2}{a_{n+2} + 1}, \frac{a_{n+3} + 2}{a_{n+3} + 1}\right)$$

and

$$C_2(n) = \frac{\log 2}{2} + \log \max\left(\frac{a_{2\lceil n/2 \rceil + 3} + 2}{a_{2\lceil n/2 \rceil + 3} + 1}, \frac{a_{2\lceil n/2 \rceil + 5} + 2}{a_{2\lceil n/2 \rceil + 5} + 1}\right).$$

Proof. Let $x = [\mathbf{a}] = [a_1, \dots]$ be an element of I and consider

$$y = [a_1, \dots, a_n, b_{n+1}, b_{n+2}, \dots],$$

with $b_{n+1} \neq a_{n+1}$; for the sake of simplicity, one can suppose that n is even. We will bound $|x - y|$ and $|f_1(x) - f_1(y)|$ with terms depending on \mathbf{a} and n only.

Since $I_n(x) = I_n(y)$, one has $|x - y| \leq |I_n(x)|$ and

$$|I_n(x)| = \frac{1}{q_n^2(\mathbf{a})} \frac{1}{1 + q_{n-1}(\mathbf{a})/q_n(\mathbf{a})} \leq \frac{1}{q_n^2(\mathbf{a})}.$$

Moreover, since

$$\begin{aligned} q_n(\mathbf{a}) &= a_n q_{n-1}(\mathbf{a}) + q_{n-2}(\mathbf{a}) \geq a_n q_{n-1}(\mathbf{a}) \\ &\geq a_n(a_{n-1} q_{n-2}(\mathbf{a}) + q_{n-3}(\mathbf{a})) \geq a_n \cdots a_3(a_2 q_1(\mathbf{a}) + q_0(\mathbf{a})) \\ &\geq a_n \cdots a_1, \end{aligned}$$

one gets

$$|x - y| \leq \frac{1}{a_1^2 \cdots a_n^2}.$$

The same reasoning can be applied to

$$f_1(x) = [a_1, a_3, \dots, a_{n-1}, a_{n+1}, \dots]$$

and

$$f_1(y) = [a_1, a_3, \dots, a_{n-1}, b_{n+1}, b_{n+3}, \dots]$$

to obtain

$$|f_1(x) - f_1(y)| \leq |I_{n/2}(f_1(x))| \leq \frac{1}{a_1^2 a_3^2 \cdots a_{n-1}^2}.$$

For the lower bound of $|x - y|$, let us remark that $I_{n+1}(x) \cap I_{n+1}(y) = \emptyset$, but the distance between $I_{n+1}(x)$ and $I_{n+1}(y)$ can be zero. However, for any fixed $j \in \mathbb{N}$, there exists a denumerably infinite number of intervals of rank $n+1+j$ in between $I_{n+1+j}(x)$ and $I_{n+1+j}(y)$, i.e. there exists a denumerably infinite number of $z \in I$ such that $z' \in I_{n+1+j}(z)$ implies $x < z' < y$ or $y < z' < x$. If $z = [\mathbf{c}]$ is such an element, one has

$$|x - y| \geq |I_{n+3}(z)| \geq \frac{1}{q_{n+3}(\mathbf{c})(q_{n+3}(\mathbf{c}) + q_{n+2}(\mathbf{c}))} \geq \frac{1}{2q_{n+3}^2(\mathbf{c})}.$$

The relations

$$\begin{aligned} q_{n+3}(\mathbf{c}) &= c_{n+3} q_{n+2}(\mathbf{c}) + q_{n+1}(\mathbf{c}) \leq (c_{n+3} + 1) q_{n+2}(\mathbf{c}) \\ &\leq (c_{n+3} + 1)(c_{n+2} q_{n+1}(\mathbf{c}) + q_n(\mathbf{c})) \leq (c_{n+3} + 1) \cdots (c_1 + 1) \end{aligned}$$

lead to

$$|I_{n+3}(z)| \geq \frac{1}{2(c_1 + 1)^2 \cdots (c_{n+3} + 1)^2}.$$

Now let

$$j_0 = \begin{cases} n+2 & \text{if } x < y \\ n+3 & \text{if } y < x \end{cases};$$

one can choose z such that $c_j = a_j$ for any $j \in \mathbb{N}$ except for the index j_0 for which $c_{j_0} = a_{j_0} + 1$, so that $z > x$ in the case $x < y$ and $z < x$ in the case $y < x$. Moreover, $I_{n+1}(z) = I_{n+1}(x) \neq I_{n+1}(y)$, so that $x < z < y$ in the case $x < y$ and $y < z < x$ in the case $y < x$. One therefore has

$$|x - y| \geq |I_{n+3}(z)| \geq \frac{1}{2(a_1 + 1)^2 \cdots (a_{n+2} + 1)^2 (a_{n+3} + 2)^2},$$

or

$$|x - y| \geq |I_{n+3}(z)| \geq \frac{1}{2(a_1 + 1)^2 \cdots (a_{n+2} + 2)^2 (a_{n+3} + 1)^2},$$

depending on the value of j_0 . Without loss of generality, one can assume that j_0 corresponds to the largest integer in such inequalities.

Now there also exists $w = [d_1, \dots]$ such that $I_{n/2+3}(w)$ lies between $I_{n/2+3}(f_1(x))$ and $I_{n/2+3}(f_1(y))$; moreover one can choose w such that $d_j = a_{2j-1}$ for any j except for one index $j_0 \in \{n/2 + 2, n/2 + 3\}$, for which $d_{j_0} = a_{2j_0-1} + 1$. One thus has

$$|f_1(x) - f_1(y)| \geq |I_{n/2+3}(w)| \geq \frac{1}{2(a_1 + 1)^2(a_3 + 1)^2 \cdots (a_{n+3} + 1)^2(a_{n+5} + 2)^2}.$$

Putting all these inequalities together and taking the logarithm, one gets

$$\frac{-2 \sum_{j=1}^{n/2} \log a_{2j-1}}{-\log 2 - 2 \sum_{j=1}^{n+3} \log(a_j + 1) - 2 \log(\frac{a_{n+3}+2}{a_{n+3}+1})} \leq \frac{\log |f_1(x) - f_1(y)|}{\log |x - y|}$$

and

$$\frac{\log |f_1(x) - f_1(y)|}{\log |x - y|} \leq \frac{-\log 2 - 2 \sum_{j=1}^{n/2+3} \log(a_{2j-1} + 1) - 2 \log(\frac{a_{n+5}+2}{a_{n+5}+1})}{-2 \sum_{j=1}^n \log a_j}.$$

□

Of course, the same reasoning can be applied to f_2 , leading to the same result.

Theorem 13. *Let $x = [\mathbf{a}]$ be an element of I and $y \in I_n(x) \setminus I_{n+1}(x)$. One has*

$$\frac{\frac{1}{n} \sum_{j=1}^{\lfloor n/2 \rfloor} \log a_{2j}}{\frac{1}{n} \sum_{j=1}^{n+3} \log(a_j + 1) + \frac{1}{n} C_1(n)} \leq \frac{\log |f_2(x) - f_2(y)|}{\log |x - y|}$$

and

$$\frac{\log |f_2(x) - f_2(y)|}{\log |x - y|} \leq \frac{\frac{1}{n} \sum_{j=1}^{\lfloor n/2 \rfloor + 3} \log(a_{2j} + 1) + \frac{1}{n} C_2(n)}{\frac{1}{n} \sum_{j=1}^n \log a_j},$$

where C_1 is defined as in Theorem 12 and

$$C_2(n) = \frac{\log 2}{2} + \log \max\left(\frac{a_{2\lfloor n/2 \rfloor + 4} + 2}{a_{2\lfloor n/2 \rfloor + 4} + 1}, \frac{a_{2\lfloor n/2 \rfloor + 6} + 2}{a_{2\lfloor n/2 \rfloor + 6} + 1}\right).$$

To obtain a generic result about the regularity of Cantor's bijection, we need a direct consequence of the ergodic theorem on continued fractions [9]. We say that a property P concerning sequences of $\mathbb{N}^{\mathbb{N}}$ holds almost everywhere if for almost every $x \in I$ (with respect to the Lebesgue measure), the sequence $\mathbf{a} \in \mathbb{N}^{\mathbb{N}}$ such that $x = [\mathbf{a}]$ satisfies P . The following result can be obtained from the main theorem of [8].

Theorem 14. *For any $k \in \mathbb{N} \cup \{0\}$, almost every sequence $\mathbf{a} \in \mathbb{N}^{\mathbb{N}}$ satisfies*

$$\frac{1}{n} \sum_{j=1}^n \log(a_j + k), \frac{1}{n} \sum_{j=1}^n \log(a_{2j} + k), \frac{1}{n} \sum_{j=1}^n \log(a_{2j-1} + k) \rightarrow \log K_k,$$

as n goes to infinity, where K_k is defined by:

$$K_k = \prod_{j=1}^{\infty} \left(1 + \frac{1}{j(j+2)}\right)^{\log(j+k)/\log 2}.$$

The seminal result $\frac{1}{n} \sum_{j=1}^n \log a_j \rightarrow \log K_0$ was proven in [6] ; K_0 is called the Khintchine's constant. Here, we will be interested in the values

$$\log K_0 \approx 0.987849056 \dots \quad \text{and} \quad \log K_1 \approx 1.409785988 \dots$$

Using Theorem 12, Theorem 13 and Theorem 14 as n goes to infinity (or equivalently as y tends to x), we get the following result.

Corollary 15. *For almost every $x \in I$, one has*

$$h_{f_1}(x), h_{f_2}(x) \in \left[\frac{\log K_0}{2 \log K_1}, \frac{\log K_1}{2 \log K_0} \right].$$

Remark 5. Let $\mathbf{a} \in \mathbb{N}^{\mathbb{N}}$ be the sequence defined by

$$a_j = \begin{cases} 2^j & \text{if } j \text{ is even} \\ 1 & \text{if } j \text{ is odd} \end{cases},$$

for any $j \in \mathbb{N}$ and set $x = [\mathbf{a}]$. It is easy to check, using Theorem 12, that for this particular point, we have $h_{f_1}(x) = 0$, so that f is a multifractal function.

Remark 6. The insiders of ergodic theory will certainly recognize the Birkhoff theorem (with the Gauss transformation, which preserves the Gauss measure and which is ergodic for this measure) behind some arguments to prove Corollary 15.

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